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Some Applications of Mass Transport to Gaussian-Type Inequalities

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Abstract

As discovered by Brenier, mapping through a convex gradient gives the optimal transport in $\mathbb{R}^n$. In the present article, this map is used in the setting of Gaussian-like measures to derive an inequality linking entropy with mass displacement by a straightforward argument. As a consequence, logarithmic Sobolev and transport inequalities are recovered. Finally, a result of Caffarelli on the Brenier map is used to obtain Gaussian correlation inequalities.

1. Introduction

Optimal mass transport can be used to derive several geometric and functional inequalities. In this paper we will use the Brenier map [6], which is known to give the optimal mass transport on $\mathbb{R}^n$. Let us recall some terminology. If μ and ν are two non-negative Borel measures on $\mathbb{R}^n$ with the same total mass, say 1, a map $T : \mathbb{R}^n \to \mathbb{R}^n$ defined μ-almost everywhere is said to *push* μ *forward* to ν (or to *transport* μ onto ν) if ν is the image of μ by T. This means that for every Borel set $B \subset \mathbb{R}^n$, $\nu(B) = \mu(T^{-1}(B))$, or equivalently, that for every non-negative Borel function $b : \mathbb{R}^n \to \mathbb{R}_+$, $\int b(y)d\nu(y) = \int b(T(x))d\mu(x)$. The idea, going back to Monge [15], is to find a map T which is "optimal" in some sense. For two Borel probability measures μ and ν on $\mathbb{R}^n$ with finite second order moment, the Wasserstein-Kantorovich distance $W(\mu, \nu)$ between μ and ν is defined by

$$W^2(\mu, \nu) := \inf_{\pi \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, d\pi(x, y),$$

where $\Gamma(\mu, \nu)$ denotes the set of Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ and ν respectively. It is a classical result that W metrizes the weak$*$ topology of the set of probability measures on $\mathbb{R}^n$ with finite second order moment. The weak$*$ topology is considered with respect to continuous functions f on $\mathbb{R}^n$

for which $f(x)/(1+|x|^2)$ is bounded. A map T pushing μ forward to ν will be said to be *optimal* if

$$W^2(\mu,\nu) = \int |x - T(x)|^2 \, d\mu(x). \tag{1}$$

While investigating optimal mass transport, BRENIER [6] found a very particular map pushing forward one probability onto another. The result, as improved by MCCANN [13], is as follow:

If μ and ν are Borel probability measures on $\mathbb{R}^n$ and μ is absolutely continuous with respect to the Lebesgue measure, then there exists a convex function ϕ such that $\nabla\phi$ pushes μ forward to ν. Furthermore, $\nabla\phi$ is uniquely determined μ-almost everywhere and, if μ and ν have finite second order moment, $\nabla\phi$ realizes the optimal transport in the sense of (1) *between μ and ν.*

Observe that ϕ is differentiable almost everywhere since it is convex. The map $T = \nabla\phi$ is usually referred to as *the* Brenier map. Some authors also speak of *monotone mass transport*. When ν is itself absolutely continuous with respect to the Lebesgue measure, then $T^{-1} = (\nabla\phi)^{-1}$ pushes ν forward to μ. If μ and ν are absolutely continuous Borel probability measures with finite second order moment, it is convenient to set $\theta(x) = \phi(x) - |x|^2/2$ so that the Brenier map takes the form $T(x) = x + \nabla\theta(x)$. Indeed (1) then becomes

$$W^2(\mu,\nu) = \int |\nabla\theta(x)|^2 \, d\mu(x).$$

If f and g are non-negative functions on $\mathbb{R}^n$ with $\int f = \int g = 1$ and $T(x) = \nabla\phi(x)$ is the Brenier map pushing $f(x)\,dx$ forward to $g(y)\,dy$, then by the definition of mass transport,

$$\int b(y)g(y)\,dy = \int b(\nabla\phi(x))f(x)dx \tag{2}$$

for every non-negative Borel function $b : \mathbb{R}^n \to \mathbb{R}_+$. Whenever the change of variable $y = \nabla\phi(x)$ is licit, (2) leads to a so-called Monge-Ampère equation:

$$f(x) = \det(\mathrm{Hess}_x\phi)\, g(\nabla\phi(x)) \tag{3}$$

However it may happen that $\nabla\phi$ exists only almost everywhere and is not differentiable in the usual sense. To handle this non-regularity we use, following MCCANN [14], the notion of Hessian in the sense of Aleksandrov. A function ϕ differentiable at $x \in \mathbb{R}^n$ is said to have a Hessian in the sense of Aleksandrov at x if there exists a symmetric linear map H such that

$$\phi(x+u) = \phi(x) + \nabla\phi(x).u + \tfrac{1}{2}Hu.u + o(|u|^2).$$

The symmetric linear map H is said to be the Hessian in the sense of Aleksandrov and will be denoted by $\mathrm{Hess}_x\phi$. A convex function ϕ admits a Hessian in the sense of Aleksandrov almost everywhere (see [8]) and, whenever it exists, $\mathrm{Hess}_x\phi$ is non-negative. We emphasize the result of MCCANN [14]:

Let μ and ν be two Borel probability measures on $\mathbb{R}^n$ with density f and g respectively, and let $\nabla\phi$ be the Brenier map pushing μ forward to ν. There exists a Borel set X of measure 1 *for μ such that for every $x \in X$ the Hessian* $\mathrm{Hess}_x\phi$ *of ϕ in the sense of Aleksandrov exists at x and equation* (3) *holds.*

The next section will be devoted to transport (Corollary 2) and logarithmic Sobolev inequalities (Corollary 1) for the Gaussian measure and for a class of Boltzmann measures. Though the language was not the same, one of the first uses of mass transport in the setting of Gaussian measure was the work of MAUREY [12]. Extending his method, BOBKOV & LEDOUX [5] used the Prékopa-Leindler inequality to derive transport and logarithmic Sobolev inequalities. Since the Prékopa-Leindler inequality is usually proved by means of mass transport, it appeared natural to ask whether the logarithmic Sobolev inequality could be proved directly by mass transport. A positive answer was given in the work of OTTO [16] where interpolation along mass transport was used to derive a new interpretation of the logarithmic Sobolev inequality. More recently, the work of OTTO & VILLANI [17] clearly showed that inequalities involving entropy can be recover by using the equations satisfied by the interpolated densities, such as Hamilton-Jacobi and Euler equations. We will reproduce their results by surprisingly simple arguments that do not involve interpolation along mass transport. Note that a similar, though less direct, method was used by BLOWER [3] to prove the transport inequality of Corollary 2.

The last section will be devoted to the study of some particular cases of the Gaussian correlation inequality conjecture

$$\gamma_n(A \cap B) \geqq \gamma_n(A)\gamma_n(B), \tag{4}$$

where γ_n is the standard n-dimensional Gaussian measure and $A, B \subset \mathbb{R}^n$ are convex symmetric sets. We will use a recent observation of CAFFARELLI [7] on the Lipschitz behavior of the Brenier map to reproduce a result of HARGÉ [10] and also to obtain some new non-symmetric extensions of a result of SIDAK [19] for (4) when B is a strip.

2. Inequalities involving entropy

Let us fix some notation. We will work on the standard Euclidean space $(\mathbb{R}^n, |\cdot|, .)$ and with the standard Gaussian measure γ_n given by $d\gamma_n(x) = (1/\sqrt{2\pi})^n\, e^{-|x|^2/2}dx$. The *entropy* with respect to a probability μ of a non-negative function $f : \mathbb{R}^n \to \mathbb{R}_+$ is defined by

$$\mathrm{Ent}_\mu(f) := \int f \log f\, d\mu - \left(\int f\, d\mu\right) \log\left(\int f\, d\mu\right),$$

and the *Fisher information* with respect to μ of a smooth non-negative f is defined by:

$$I_\mu(f) := \int \frac{|\nabla f|^2}{f} d\mu = 4 \int \left|\nabla\sqrt{f}\right|^2 d\mu.$$

The Gaussian logarithmic Sobolev inequality discovered by GROSS [9] states that, for every smooth non-negative function f on $\mathbb{R}^n$,

$$\mathrm{Ent}_{\gamma_n}(f) \leqq \tfrac{1}{2} I_{\gamma_n}(f).$$

As an appetizer, let us give a simple proof of this inequality.

Proof. We may assume $\int f\, d\gamma_n = 1$. Let $\nabla\phi$ be the Brenier map between $f\, d\gamma_n$ and γ_n. Write $\theta(x) = \phi(x) - \frac{1}{2}|x|^2$ so that $\nabla\phi(x) = x + \nabla\theta(x)$. Denoting by I the identity matrix, we have $I + \mathrm{Hess}_x\theta \geqq 0$. The Monge-Ampère equation holding $f\,d\mu$-a.e. is:

$$f(x)e^{-|x|^2/2} = \det(I + \mathrm{Hess}_x\theta)\, e^{-|x+\nabla\theta(x)|^2/2}.$$

After taking the logarithm, we have:

$$\begin{aligned}
\log f(x) &= -\tfrac{1}{2}|x + \nabla\theta(x)|^2 + \tfrac{1}{2}|x|^2 + \log\det(I + \mathrm{Hess}_x\theta)\\
&= -x.\nabla\theta(x) - \tfrac{1}{2}|\nabla\theta(x)|^2 + \log\det(I + \mathrm{Hess}_x\theta)\\
&\leqq -x.\nabla\theta(x) - \tfrac{1}{2}|\nabla\theta(x)|^2 + \Delta\theta(x),
\end{aligned}$$

where we used $\log(1+t) \leqq t$ whenever $1 + t \geqq 0$. We integrate with respect to $f\,d\gamma_n$:

$$\int f \log f\, d\gamma_n \leqq \int f[\Delta\theta - x.\nabla\theta]d\gamma_n - \int \tfrac{1}{2}|\nabla\theta|^2\, f\,d\gamma_n.$$

By integration by parts we get:

$$\begin{aligned}
\int f\log f\, d\gamma_n &\leqq -\int \nabla\theta.\nabla f\, d\gamma_n - \int \frac{1}{2}|\nabla\theta|^2 f\, d\gamma_n\\
&= -\int \frac{1}{2}\left|\sqrt{f}\nabla\theta(x) + \frac{\nabla f(x)}{\sqrt{f}}\right|^2 d\gamma_n(x) + \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\gamma_n\\
&\leqq \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\gamma_n.
\end{aligned}$$

The careful reader may have noticed that the proof above is not completely correct. Indeed, we assumed that the trace of the Hessian in the sense of Aleksandrov coincided with the Laplacian in the sense of distribution. This is not true in general. However there is an inequality (going in the right direction for us!) linking these two notions. For a convex function ϕ, possibly perturbed by a quadratic polynomial (for instance $\theta(x) = \phi(x) - |x|^2/2$) the Laplacian in the sense of Aleksandrov is the trace of the $\mathrm{Hess}_x\phi$. It is Borel function defined almost everywhere on the domain of ϕ (see the Appendix) and we will denote it by $\Delta_A\phi$. We introduce the (interior) domain of a convex function ϕ as the convex open set given by $\mathrm{Dom}(\phi) := \mathrm{int}\{\phi < +\infty\}$.

Lemma 1. *Let ϕ be a convex function on $\mathbb{R}^n$ with domain U. Then for every smooth, compactly supported in U, non-negative function f, the following holds:*

$$\int f\,\Delta_A\phi \leqq -\int \nabla f.\nabla\phi\ .$$

The proof of the lemma will be given in the Appendix. The lemma obviously also applies to the case of a function $\theta(x) = \phi(x) - |x|^2/2$ where ϕ is convex. Still, there is a gap in the argument above, since f need not be supported inside the domain of ϕ. Indeed, assume f is compactly supported and let $T(x) = \nabla\phi(x)$ be the Brenier map pushing $f\,d\gamma_n$ forward to γ_n. Then $\nabla\phi$ necessarily tends to ∞ at many points of the boundary of the support of f. However this problem is easy to remove by approximation, as we will see.

Following the same method, it is possible to prove TALAGRAND's inequality [21] for the Wasserstein-Kantorovich distance. But it would then be preferable to transport γ_n on $f d\gamma_n$. As a matter of fact, transport and logarithmic Sobolev inequalities follow from a general inequality holding for a wide class of measures. This inequality is closely related to the results and the comments of the work of OTTO & VILLANI [17]. They used interpolation along mass transport. Our proof, inspired by their work, is direct and much simpler. It is more or less the same as the one presented above in the case of the Gaussian measure, and does not require any partial-differential-equation argument. For a real c and a real symmetric matrix A we shall write $A \geqq c$ for $A \geqq cI$.

Theorem 1. *Let μ be a probability measure on $\mathbb{R}^n$ of the form $d\mu(x) = e^{-V(x)}dx$, where V is a twice differentiable function satisfying* Hess $V \geqq c$ *for some $c \in \mathbb{R}$. Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be non-negative compactly supported functions, with f $\mathcal{C}^1$ and $\int f\,d\mu = \int g\,d\mu$. If $T(x) = x + \nabla\theta(x)$ is the Brenier map pushing $f\,d\mu$ forward to $g\,d\mu$, then*

$$\mathrm{Ent}_\mu(g) \geqq \mathrm{Ent}_\mu(f) + \int \nabla f.\nabla\theta\,d\mu + \frac{c}{2}\int |\nabla\theta|^2 f d\mu.$$

Proof. We can assume that $\int f d\mu = \int g d\mu = 1$. Since f and g are compactly supported, $\nabla\theta$ remains bounded on the support of f and thus the support of f is contained in $\mathrm{Dom}(\theta)$. The Monge-Ampère equation holding $f d\mu$-a.e. is

$$f(x)e^{-V(x)} = g(T(x))\,e^{-V(T(x))}\,\det(I + \mathrm{Hess}_x\theta).$$

After taking the logarithm, we get

$$\log g(T(x)) = \log f(x) + V(x + \nabla\theta(x)) - V(x) - \log\det(I + \mathrm{Hess}_x\theta).$$

As before, we use the fact that $\log\det(I + \mathrm{Hess}_x\theta) \leqq \Delta_A\theta(x)$. By integral Taylor expansion, we get

$$V(x + \nabla\theta(x)) - V(x) \geqq \nabla V(x).\nabla\theta(x) + \frac{c}{2}|\nabla\theta(x)|^2.$$

This implies that $f d\mu$-a.e.

$$\log g(T(x)) \geqq \log f(x) + \nabla V(x).\nabla\theta(x) - \Delta_A\theta(x) + \frac{c}{2}|\nabla\theta(x)|^2.$$

Now, we integrate with respect to $f d\mu$. First, observe that by the definition of the transport,

$$\int (\log g(T(x)))f(x)d\mu(x) = \int (\log g(x))g(x)d\mu(x) = \mathrm{Ent}_\mu(g).$$

Integration by parts (Lemma 1) gives:

$$\int [\nabla V(x).\nabla\theta(x) - \Delta_A\theta(x)] f(x) d\mu(x) \geqq \int \nabla\theta(x).\nabla f(x) d\mu.$$

This ends the proof of the theorem. □

We present now some direct consequences of Theorem 1. We start with the Bakry-Emery [1] logarithmic Sobolev inequality on $\mathbb{R}^n$.

Corollary 1 (Logarithmic Sobolev inequality [1]**).** *Let μ be a probability measure on $\mathbb{R}^n$ of the form $d\mu(x) = e^{-V(x)}dx$, where V is a twice differentiable function satisfying* Hess $V \geqq c$ *for some $c > 0$. Then, for every non-negative smooth function f,*

$$\mathrm{Ent}_\mu(f) \leqq \frac{1}{2c} I_\mu(f).$$

Proof. We can assume that f is compactly supported. Let g be any non-negative compactly supported function with $\int g\, d\mu = \int f\, d\mu = 1$, say. Apply Theorem 1 to f and to g and use the fact that

$$\begin{aligned} \nabla f.\nabla\theta + \frac{c}{2} f|\nabla\theta|^2 &= \frac{c}{2} f \left| \frac{\nabla f}{cf} + \nabla\theta \right|^2 - \frac{1}{2c} \frac{|\nabla f|^2}{f} \\ &\geqq -\frac{1}{2c} \frac{|\nabla f|^2}{f}. \end{aligned}$$

This gives

$$\mathrm{Ent}_\mu(g) + \frac{1}{2c} I_\mu(f) \geqq \mathrm{Ent}_\mu(f).$$

Now if $g \to 1$, we have $\mathrm{Ent}_\mu(g) \to 0$, and this finishes the proof. □

We now derive a generalization of TALAGRAND's transport inequality [21] recently obtained by BLOWER [3], OTTO & VILLANI [17], BOBKOV & LEDOUX [5].

Corollary 2 (Transport inequality**).** *Let μ be a probability measure on $\mathbb{R}^n$ of the form $d\mu(x) = e^{-V(x)}dx$, where V is a twice differentiable function satisfying* Hess $V \geqq c$ *for some $c > 0$. Then, for every non-negative function g with $\int g d\mu =$* 1,

$$W^2(\mu, g d\mu) \leqq \frac{2}{c} \mathrm{Ent}_\mu(g).$$

Proof. Assume first that g is compactly supported. Formally the result follows by applying theorem 1 to $f \equiv 1$. Let f_n be a sequence of non-negative smooth compactly supported functions such that $f_n \to 1$ with $\int |\nabla\sqrt{f}_n|^2\, d\mu \to 0$ and $f_n \leqq 2$. Normalize the sequence so that $\int f_n\, d\mu = 1$. Let $T_n(x) = x + \nabla\theta_n(x)$ be the Brenier map pushing $f_n\, d\mu$ forward to $g\, d\mu$ and apply Theorem 1. We know that $\mathrm{Ent}_\mu(f_n) \to 0$ and $\int |\nabla\theta|^2 f_n d\mu = W^2(f_n d\mu, g d\mu) \longrightarrow W^2(\mu, g d\mu)$. We have $I_\mu(f_n) \to 0$ and by Hölder's inequality,

$$\left| \int \nabla\theta_n .\nabla f_n d\mu \right| \leqq \sqrt{\int |\nabla\theta_n|^2 f_n d\mu \int \frac{|\nabla f_n|^2}{f_n} d\mu} = W(f_n d\mu, g d\mu)\sqrt{I_\mu(f_n)},$$

we deduce that $\int \nabla\theta_n . \nabla f_n \to 0$. Thus at the limit we get

$$\mathrm{Ent}_\mu(g) \geqq \frac{c}{2} W^2(\mu, g d\mu).$$

Again by approximation the result extends to non-compactly supported g's. $\square$

Finally, we recover an inequality linking the entropy, the Fisher information and the Wasserstein-Kantorovich distance, and referred to as "HWI inequality" in the work of Otto and Villani.

Corollary 3 (HWI inequality [16,17]). *Let μ be a probability measure on $\mathbb{R}^n$ of the form $d\mu(x) = e^{-V(x)}dx$, where V is a twice differentiable function satisfying* Hess $V \geqq c$ *for some $c \in \mathbb{R}$. Then, for every non-negative compactly supported smooth function f with $\int f d\mu = 1$,*

$$\mathrm{Ent}_\mu(f) \leqq W(\mu, f d\mu)\sqrt{I_\mu(f)} - \frac{c}{2} W^2(\mu, f d\mu).$$

Proof. Apply Theorem 1 to f and to some non-negative compactly supported g, and use again

$$\int \nabla\theta . \nabla f d\mu \geqq -\sqrt{\int |\nabla\theta|^2 f d\mu \int \frac{|\nabla f|^2}{f} d\mu} = -W(f d\mu, g d\mu)\sqrt{I_\mu(f)}.$$

Thus we have

$$\mathrm{Ent}_\mu(g) \geqq \mathrm{Ent}_\mu(f) - W(f d\mu, g d\mu)\sqrt{I_\mu(f)} + \frac{c}{2} W^2(f d\mu, g d\mu).$$

When $g \to 1$ we get the result. $\square$

Otto and Villani observed that Corollary 1 implies Corollary 2 (actually without any convexity condition on V), and that Corollary 2 and Corollary 3 imply Corollary 1.

Let us comment briefly on the equality cases in these inequalities when the measure μ is the Gaussian measure γ_n. In this case the main inequality that was used, *assuming* the Brenier map was smooth (i.e., θ was $\mathcal{C}^2$), was

$$\log\det(I + \mathrm{Hess}\,\theta) \leqq \Delta\theta.$$

Equality cases require $\mathrm{Hess}\,\theta = 0$ or, equivalently, $\nabla\theta \equiv u$ for some $u \in \mathbb{R}^n$. Thus the Brenier map needs to be a translation. In order to prove the logarithmic Sobolev or the transport inequality, we take f or g equal to 1 and in these cases we can easily check that, if $\nabla\theta \equiv u$, there is equality at every step. So we can guess that the equality cases in these two inequalities are given by functions f for which $f d\gamma_n$ is a translation of γ_n, meaning also functions f that are exponential : $f(x) = c\, e^{x.u}$. Making this argument rigorous would require too many technical results on the behavior of the Brenier map and on the regularity of the solutions of the Monge-Ampère equation, and would not simplify the existing proofs. However it helps to understand the picture: exponential functions arise naturally when translating the Gaussian measure.

3. Applications of a result of Caffarelli

This section deals with non-negative *log-concave* functions f on $\mathbb{R}^n$ (meaning $\log f$ is concave) and convex sets in $\mathbb{R}^n$. In this particular setting, where f and g have convex supports, regularity results of CAFFARELLI (see [7] and references therein) ensure that the Brenier map between f and g is smooth under weak assumptions on f and g. We will always assume that we are in a situation were these regularity results apply.

Recently, CAFFARELLI [7] made a crucial observation on the behavior of the Brenier map. He proved that given a Gaussian probability measure ρ and a probability measure μ with log-concave density with respect to ρ, the Brenier map pushing forward ρ onto μ is a contraction (i.e., 1-Lipschitz). We can even have a quantitative estimate. Let μ be a probability measure of the form $d\mu(x) = e^{-V(x)}dx$ where V is a smooth function with $\mathrm{Hess}_x V \geqq c > 0$. Then if $\nabla\phi$ is the Brenier map pushing γ_n forward to μ,we have

$$0 \leqq \mathrm{Hess}\,\phi \leqq \frac{1}{\sqrt{c}} \cdot \tag{5}$$

Let us mention a direct consequence of this estimate. It is possible to transport isoperimetric, logarithmic Sobolev or Poincaré (this was mentioned by Caffarelli) inequalities from the Gaussian measure to the measure μ. This allows us to rewrite the inequalities of BAKRY & EMERY [1] (Corollary 1) and of BAKRY & LEDOUX [2] for the measure μ. For instance, the strongest of these inequalities, the Gaussian isoperimetric inequality, states in its functional form [4] that, for any smooth compactly supported function $f : \mathbb{R}^n \to [0, 1]$,

$$\mathcal{U}\left(\int f d\gamma_n\right) \leqq \int \sqrt{\mathcal{U}^2(f) + |\nabla f|^2}\, d\gamma_n. \tag{6}$$

Here $\mathcal{U}(a)$ is the Gaussian surface measure of any half-space of Gaussian measure $a \in [0, 1]$. In other words, if

$$\Phi(t) = \int_{-\infty}^{t} e^{-u^2/2} \frac{du}{\sqrt{2\pi}},$$

then $\mathcal{U} = \Phi' \circ \Phi^{-1}$. Now, for μ as above, we deduce from (6) that for every smooth compactly supported function $g : \mathbb{R}^n \to [0, 1]$,

$$\mathcal{U}\left(\int g d\mu\right) \leqq \int \sqrt{\mathcal{U}^2(g) + \frac{1}{c}|\nabla g|^2}\, d\mu. \tag{7}$$

Indeed, for given g apply (6) to $f(x) = g(\nabla\phi(x))$ and use (5) together with the definition of mass transport. Inequality (7) appeared first in [2]. It is known that Corollary 1 is a consequence of (7) (see [11] for a written proof).

The applications of the result of Caffarelli we would like to outline concern the Gaussian correlation inequality (see [18] for references and partial results). This celebrated conjecture claims that, for two symmetric convex sets A and B,

$$\gamma_n(A \cap B) \geqq \gamma_n(A)\gamma_n(B). \tag{8}$$

An equivalent way of stating the Gaussian correlation inequality is to say that, for every even non-negative log-concave functions f and g,

$$\int fg\, d\gamma_n \geqq \left(\int f d\gamma_n\right)\left(\int g d\gamma_n\right).$$

One of the best results in this direction is due to HARGÉ [10] who proved that (8) holds when A is any symmetric convex set and B is an ellipsoid, possibly degenerate. Ellipsoids are assumed to be centered at the origin. By degenerate ellipsoid, we simply mean a limit of ellipsoids, as for instance a symmetric strip,

$$B_u = \{x \in \mathbb{R}^n\,;\, |x.u| \leqq 1\}.$$

In particular, Hargé's inequality generalized a result of SIDAK [19] who proved (8) when B is a symmetric strip. The proof of Hargé is based on a semi-group method involving a modified Ornstein-Uhlenbeck semi-group. We would like to present here a simple proof that uses the Brenier map and the result of Caffarelli. The ellipsoid B is given by a positive symmetric linear map H for which $B = \{|Hx| \leqq 1\}$. We introduce the Gaussian probability measure ρ given by

$$d\rho(x) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det H}}\, e^{-H^{-1}x.x}\, dx.$$

After the change of variable $y = \sqrt{H^{-1}}x$, Hargé's inequality becomes:

$$\rho(A \cap B_2^n) \geqq \rho(A)\rho(B_2^n),$$

for every convex symmetric set A and for the Euclidean unit ball B_2^n. Let ρ_A be the normalized restriction of the measure ρ to the set A, and T be the Brenier map between ρ and ρ_A. By the result of Caffarelli we know that T is a contraction. And by symmetry of the situation, $T(-x) = -T(x)$. In particular $T(0) = 0$. Hence

$$T(B_2^n) \subset B_2^n.$$

But then

$$\frac{\rho(A \cap B_2^n)}{\rho(A)} = \rho_A(B_2^n) = \rho(T^{-1}(B_2^n)) \geqq \rho(B_2^n),$$

which is precisely the desired correlation inequality.

As the reader may have noticed, the symmetry of A was only used to give $T(0) = 0$. This allows us to obtain some (weaker) statement for non-symmetric convex bodies. For a convex body K let us denote by $\mathrm{Iso}(K)$ the group of isometries leaving K (globally) invariant and by $\mathrm{Fix}(K)$ the subset of K,

$$\mathrm{Fix}(K) := \{x \in K\,;\; r(x) = x \quad \forall r \in \mathrm{Iso}(K)\}.$$

Of course, if K is symmetric, $\mathrm{Fix}(K) = \{0\}$, since $x \to -x$ has only 0 as fixed point. But, for instance, for the regular simplex Δ, we also have $\mathrm{Fix}(\Delta) = \{0\}$. We have:

Proposition 1. *Let A be a convex body with* $\mathrm{Fix}(A) = \{0\}$*. Then*

$$\gamma_n(A \cap B_2^n) \geqq \gamma_n(A)\gamma_n(B_2^n).$$

Proof. We have to prove that, if T is the Brenier map pushing γ_n forward to γ_A, the normalized restriction of the Gaussian measure to A, then $T(0) = 0$.

Claim. *Let T be the Brenier map pushing γ_n forward to γ_A (the normalized restriction of the Gaussian measure to A). Then for an isometry r the map $T_r(x) := r^{-1}T(r(x))$ is the Brenier map pushing γ_n forward to $\gamma_{r^{-1}(A)}$.*

Proof of the claim. The Gaussian measure is invariant under isometries and thus T_r clearly pushes γ_n forward to $\gamma_{r^{-1}(A)}$. We need to check that T_r is the Brenier map, i.e., that T_r is the gradient of a convex function. If $T = \nabla\phi$, introduce $\phi_r(x) = \phi(r(x))$. Then ϕ_r is convex and $\nabla\phi_r(x) = r^*\nabla\phi(r(x)) = r^{-1}T(r(x)) = T_r(x)$. □

We finish the proof of the proposition. The previous claim combined with the uniqueness of the Brenier map ensures that, for $r \in \mathrm{Iso}(A)$, we have $T_r = T$ and $T(r(x)) = r(T(x))$. This gives $T(0) = r(T(0))$ and therefore $T(0) \in \mathrm{Fix}(A)$. And by assumption this implies $T(0) = 0$ and finishes the proof. □

The next non-symmetric result is related to a statement of SZAREK & WERNER [20]. They proved that, for every convex body A of $\mathbb{R}^n$ and for every strip B symmetric with respect to the Gaussian barycenter of A, $\gamma_n(A\cap B) \geqq \gamma_n(A)\gamma_n(B)$. Our result will deal with "medians" rather than barycenters. First we treat the one-dimensional case. We introduce the median of a continuous function $f : \mathbb{R} \to \mathbb{R}_+$ with respect to the Gaussian measure γ as the set

$$\mathrm{med}_\gamma(f) := \left\{x \in \mathbb{R}\,; \int_x^{+\infty} f\,d\gamma = \frac{1}{2}\int_{\mathbb{R}} f\,d\gamma\right\}.$$

For a log-concave function, the median is obviously unique.

Proposition 2. *Let f be a non-negative log-concave function on $\mathbb{R}$ with* $\mathrm{med}(f) =: x_0$*. Then, for every $\alpha \geqq 0$,*

$$\int_{x_0-\alpha}^{x_0+\alpha} f\,d\gamma \geqq \gamma([-\alpha,\alpha])\int_{\mathbb{R}} f\,d\gamma \geqq \gamma([x_0-\alpha, x_0+\alpha])\int_{\mathbb{R}} f\,d\gamma.$$

Proof. Let T be the Brenier map pushing γ forward to $f\,d\gamma/(\int f\,d\gamma)$. Mass transport in dimension one preserves the median. Indeed, we have by definition,

$$\int_{T(0)}^{\infty} f\,d\gamma = \left(\int_{\mathbb{R}} f\,d\gamma\right)\int_0^{\infty} d\gamma = \frac{1}{2}\left(\int_{\mathbb{R}} f\,d\gamma\right).$$

Therefore $T(0) = x_0$. This implies, since T is a contraction, that $T([-\alpha,\alpha]) \subset [x_0-\alpha, x_0+\alpha]$ which in turn gives

$$\int_{x_0-\alpha}^{x_0+\alpha} f\,d\gamma \Big/ \left(\int_{\mathbb{R}} f\,d\gamma\right) = \gamma(T^{-1}[x_0-\alpha, x_0+\alpha]) \geqq \gamma([-\alpha,\alpha]).$$

And by the log-concavity of the Gaussian measure, $\gamma([-\alpha,\alpha]) \geqq \gamma([x_0-\alpha, x_0+\alpha])$. □

Remark 1. An equivalent way of stating Proposition 2 is to say that if f is a non-negative log-concave function on $\mathbb{R}$ with $\mathrm{med}_\gamma(f) =: x_0$, then for every *even* non-negative log-concave function g,

$$\int_{\mathbb{R}} f(x)g(x-x_0)\,d\gamma(x) \geqq \int_{\mathbb{R}} f\,d\gamma \int_{\mathbb{R}} g\,d\gamma \geqq \int_{\mathbb{R}} f\,d\gamma \int_{\mathbb{R}} g(x-x_0)\,d\gamma(x).$$

Indeed, with the same notation as in the above proof, we have

$$\int_{\mathbb{R}} f(x)g(x-x_0)\,d\gamma(x) \Big/ \Big(\int_{\mathbb{R}} f\,d\gamma \Big) = \int_{\mathbb{R}} g(T(x)-x_0)\,d\gamma(x).$$

Since T is a (increasing) contraction, we can write $T(x) = x+\theta'$ where θ is concave and satisfies $\theta'(0) = x_0$. Then, $T(x)-x_0 = x-x_0+\theta'(x) = x+(\theta'(x)-\theta'(0))$. The function θ' is non-increasing, therefore $T(x)-x_0$ is bigger than x if x is negative, and smaller than x if x is non-negative. But the function g, being even and log-concave is non-decreasing on $\mathbb{R}_-$ and non-increasing on $\mathbb{R}_+$. This implies that $g(T(x)-x_0) \geqq g(x)$, which gives the result.

Proposition 2 leads to a generalization of the result of Sidak. For a convex body K in $\mathbb{R}^n$ and a direction u (which means for us a vector u with $|u| = 1$), the Gaussian median $\mathrm{med}_{\gamma_n}(K,u)$ of K in the direction u is the unique real t satisfying

$$\gamma_n(K \cap \{x.u \geqq t\}) = \tfrac{1}{2}\gamma_n(K).$$

Corollary 4. *Let K be a convex body of $\mathbb{R}^n$. Fix a direction u and set $t_0 := \mathrm{med}_{\gamma_n}(K,u)$. Then, for every strip C of the form $C = \{t_0-\alpha \leqq |x.u| \leqq t_0+\alpha\}$,*

$$\gamma_n(K\cap C) \geqq \gamma_n(K)\gamma([-\alpha,\alpha]) \geqq \gamma_n(K)\gamma_n(C).$$

Proof. Introduce the section function $f_u(t) := \gamma_{n-1}(K\cap(u^\perp+tu))$. By the Prékopa-Leindler inequality, the function f_u is log-concave on $\mathbb{R}$. We have

$$\gamma_n(K) = \int_{\mathbb{R}} f_u(t)d\gamma(t)$$

and

$$\gamma_n(K\cap\{x.u \geqq t_0\}) = \int_{t_0}^{+\infty} f_u(t)\,d\gamma(t).$$

In particular f_u has t_0 as Gaussian median. Apply Proposition 2 and observe that

$$\int_{t_0-\alpha}^{t_0+\alpha} f_u(t)\,d\gamma(t) = \gamma_n(K\cap C). \qquad \square$$

Remark 2. The use of the median is natural if nice correlation inequalities are desired, in the spirit of the results of Szarek and Werner. However it was not crucial in the proof. Indeed, let f be a non-negative log-concave function on $\mathbb{R}$ and fix $t \in \mathbb{R}$. Let $k \in [0,1]$ be such that

$$\int_{-\infty}^{t} f\,d\gamma = k\int_{\mathbb{R}} f\,d\gamma.$$

Then, by definition of mass transport, the Brenier map T pushing γ forward to $f\,d\gamma/(\int f\,d\gamma)$ satisfies $\Phi(T^{-1}(t)) = k$, where Φ denotes as before the Gaussian distribution function. Using the fact that T is a contraction leads to the following inequality: for $\alpha \in \mathbb{R}$,

$$\int_{t-\alpha}^{t+\alpha} f\,d\gamma \geqq \gamma([\Phi^{-1}(k)-\alpha, \Phi^{-1}(k)+\alpha]) \int_{\mathbb{R}} f\,d\gamma. \tag{9}$$

It is then possible to derive a minorant for the intersection of a convex body in $\mathbb{R}^n$ with a strip. It must be noted that when applied to the barycenter inequality (9) might be weaker than the one of Szarek and Werner. For instance, if the barycenter is at the origin and we apply (9) for $t = 0$, the term $\gamma([\Phi^{-1}(k)-\alpha, \Phi^{-1}(k)+\alpha])$ on right-hand side is certainly smaller than $\gamma([-\alpha, \alpha])$.

4. Appendix: proof of Lemma 1

For a convex function ϕ with domain U, the function $\Delta_A\phi$ defined almost-everywhere is non-negative and $\Delta\phi$ (in the sense of distributions) is a non-negative measure on U. Recall that, by Rademacher's theorem (see [8]), the function ϕ has a gradient $\nabla\phi$ almost everywhere, and that $\nabla\phi$ is equal to the derivative of ϕ in the sense of distribution. Therefore Lemma 1 is equivalent to

$$\Delta_A\phi \leqq \Delta\phi.$$

We are led to prove that for any smooth, compactly supported in U, non-negative function f,

$$\int f\,\Delta_A\phi \leqq \int \phi\Delta f. \tag{10}$$

For $h > 0$ introduce, for any function g on $\mathbb{R}^n$,

$$g_h(x) := \frac{g(x+he_1)+g(x-he_1)-2g(x)}{h^2}.$$

Then, denoting by $(\partial_A\phi)_{11} := \mathrm{Hess}_x\phi(e_1).e_1$ the second derivative in the sense of Aleksandrov (where it exists) of ϕ in the direction e_1, we have, almost everywhere,

$$\lim_{h\to 0} \phi_h(x) = (\partial_A\phi)_{11}(x).$$

But obviously,

$$\int \phi_h f = \int \phi f_h.$$

Therefore by Fatou's lemma since $\phi_h \geqq 0$,

$$\int f(\partial_A\phi)_{11} \leqq \liminf \int \phi f_h = \int \phi(\partial f)_{11}.$$

We have the same inequalities for the other directions and this leads to (10).

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